Objectives

- To develop methods for determining the moment of inertia for an area.
- To understand the use of the parallel axis theorem.

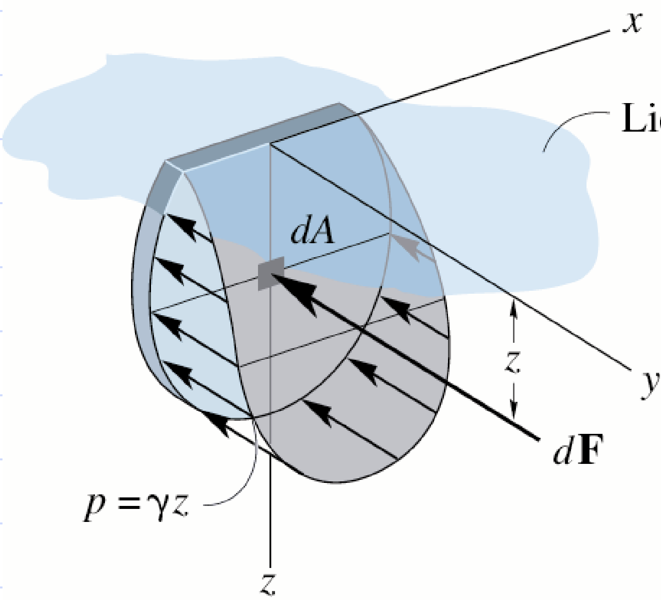


Figure 10.01

Called "second moment" of the area or the moment of inertia

$$\int x^2 dA$$

or

$$\int y^2 dA$$

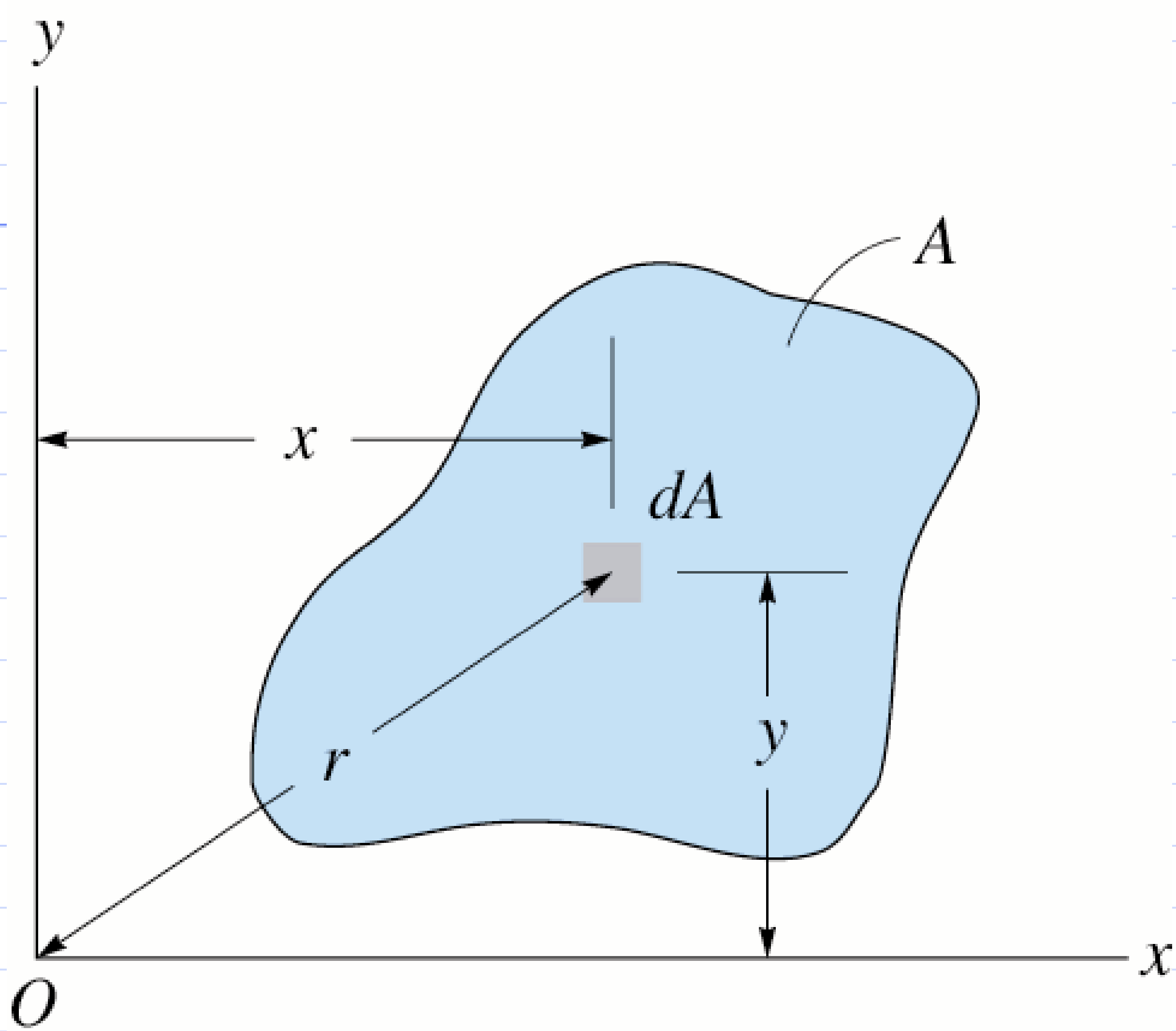


Figure 10.02

$$I_x = \int y^2 dA$$

$$I_y = \int x^2 dA$$

and

$$J_o = \int r^2 dA = I_x + I_y$$

Polar Moment of Inertia

$$J_o = \int r^2 dA = I_x + I_y$$

$$r^2 = x^2 + y^2$$

$$J_o, I_x, I_y > 0$$

Units:

$(\text{length})^4$

m^4

mm^4

ft^4

in^4

Parallel Axis Theorem

$$I_x = \int y^2 dA$$

$$I_x = \int (y' + d_y)^2 dA$$

$$I_x = \int (y'^2 + 2d_y y' + d_y^2) dA$$

$$I_x = \int y'^2 dA + 2d_y \int y' dA + d_y^2 \int dA$$

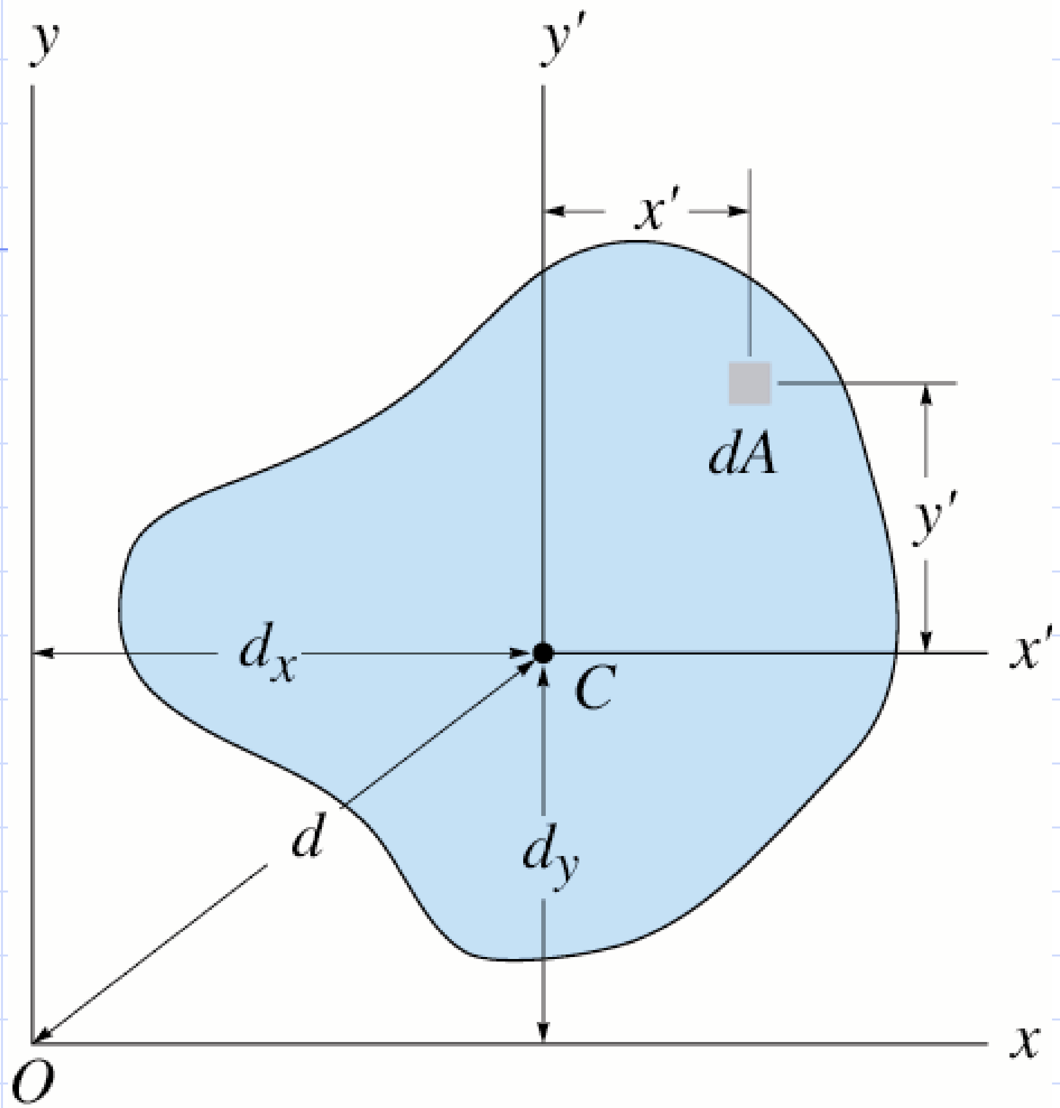


Figure 10.03

Parallel Axis Theorem

$$I_x = \int y'^2 dA + 2d_y \int y' dA + d_y^2 \int dA$$

$$\int y' dA = 0 \quad \textit{centroid}$$

$$I_x = \bar{I}_{x'} + A d_y^2$$

Parallel Axis Theorem:

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$I_y = \bar{I}_{y'} + Ad_x^2$$

$$J_O = \bar{J}_C + Ad^2$$

Parallel Axis Theorem:

The moment of inertia of an area about an axis is equal to the moment of inertia of the area about a parallel axis passing through the centroid plus the product of the area and the square of the perpendicular distance

Radius of Gyration of an Area

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

Procedure for Analysis

Case I

- Specify a differential element dA with length parallel to axis.
- Apply appropriate integration formula.

$$I_y = \int x^2 dA$$

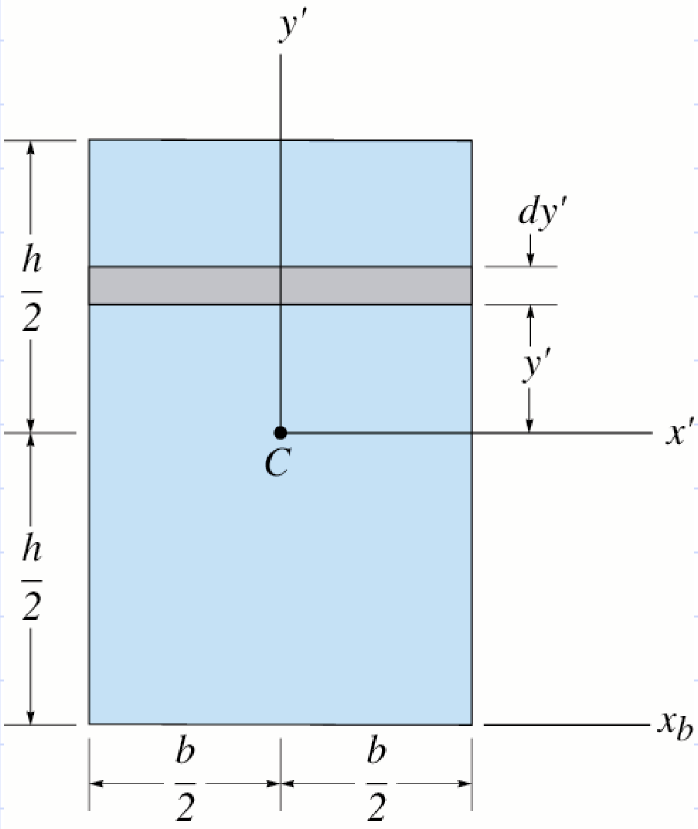


Figure 10.05

Procedure for Analysis

Case II

- Specify a differential element dA with length perpendicular to axis.
- Calculate ***I*** about centroidal axis and apply parallel axis theorem.

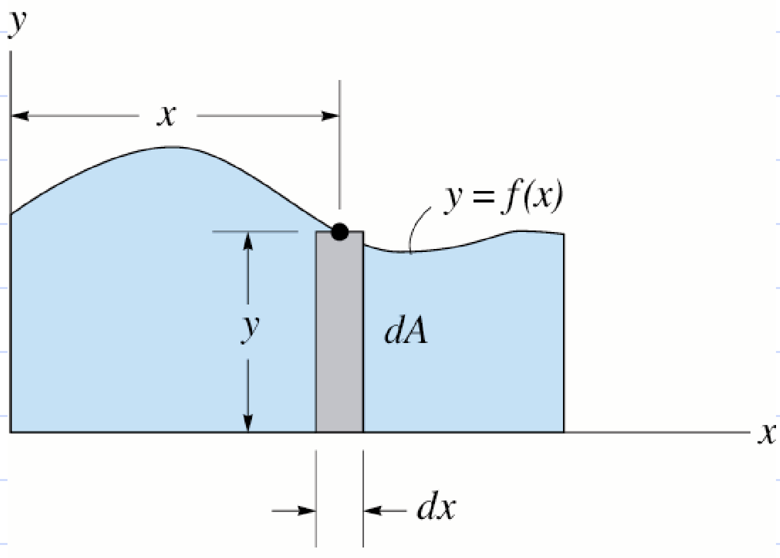
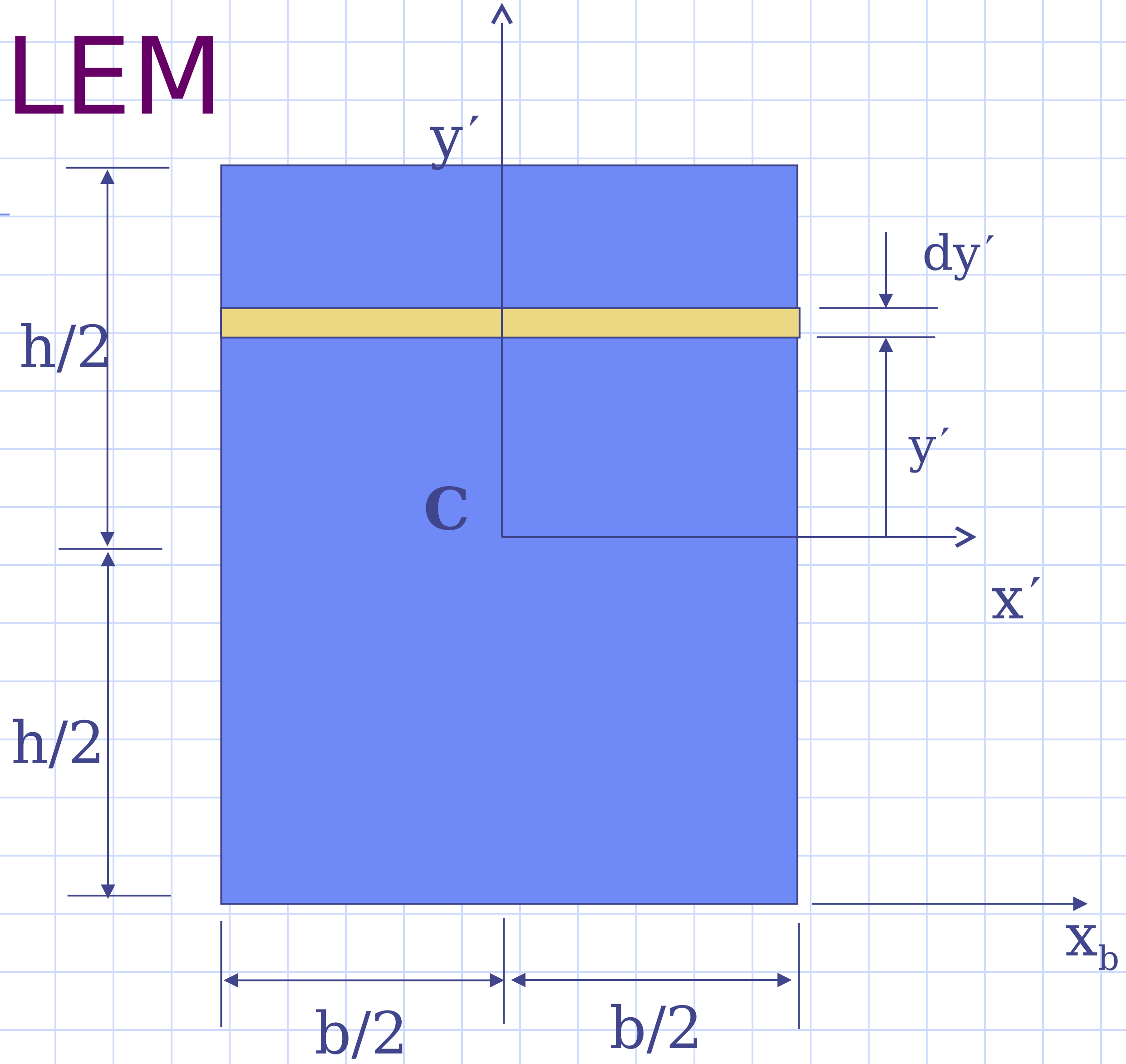


Figure 10.04

PROBLEM



Through Centroidal Axis:

$$\bar{I}_{x'} = \int_A y'^2 dA = \int_{-h/2}^{+h/2} y'^2 \, b \, dy'$$

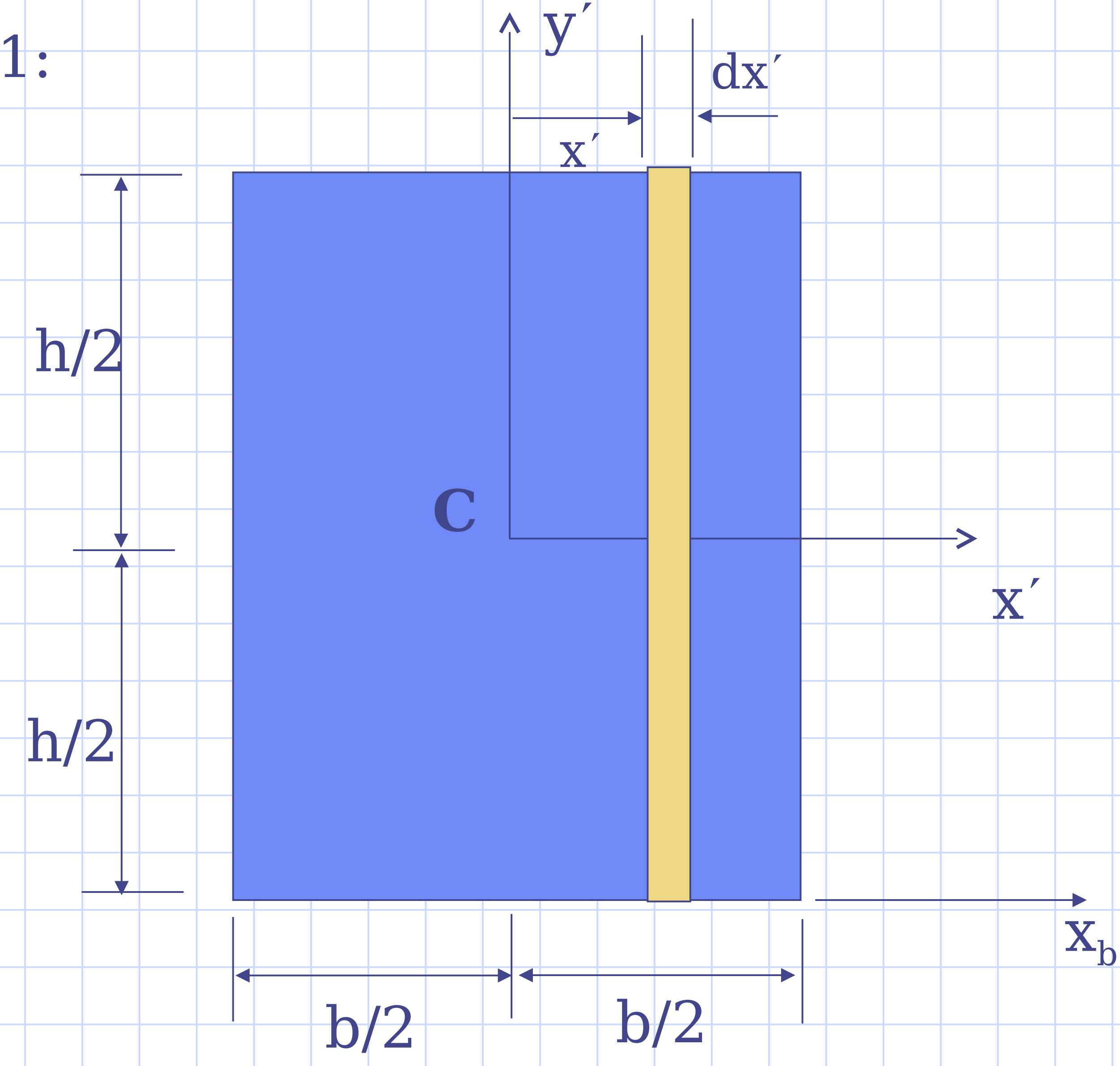
$$\bar{I}_{x'} = b \int_{-h/2}^{+h/2} y'^2 dy' = \frac{1}{12} bh^3$$

hrough Axis Passing Thorough Base

$$\bar{I}_{x_b} = \bar{I}_{x'} + A d_y^2$$

$$\bar{I}_{x_b} = \frac{1}{12} b h^3 + b h \left(\frac{h}{2}\right)^2 = \frac{1}{3} b h^3$$

Example 10-1:



$$\bar{I}_{y'} = \int_A x'^2 \, dA = \int_{-b/2}^{+b/2} x'^2 \, h \, dx'$$

$$\bar{I}_{y'} = h \int_{-b/2}^{+b/2} x'^2 \, dx' = \frac{1}{12} h b^3$$

$$J_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}\left(b h^3 + h b^3\right)$$

PROBLEM

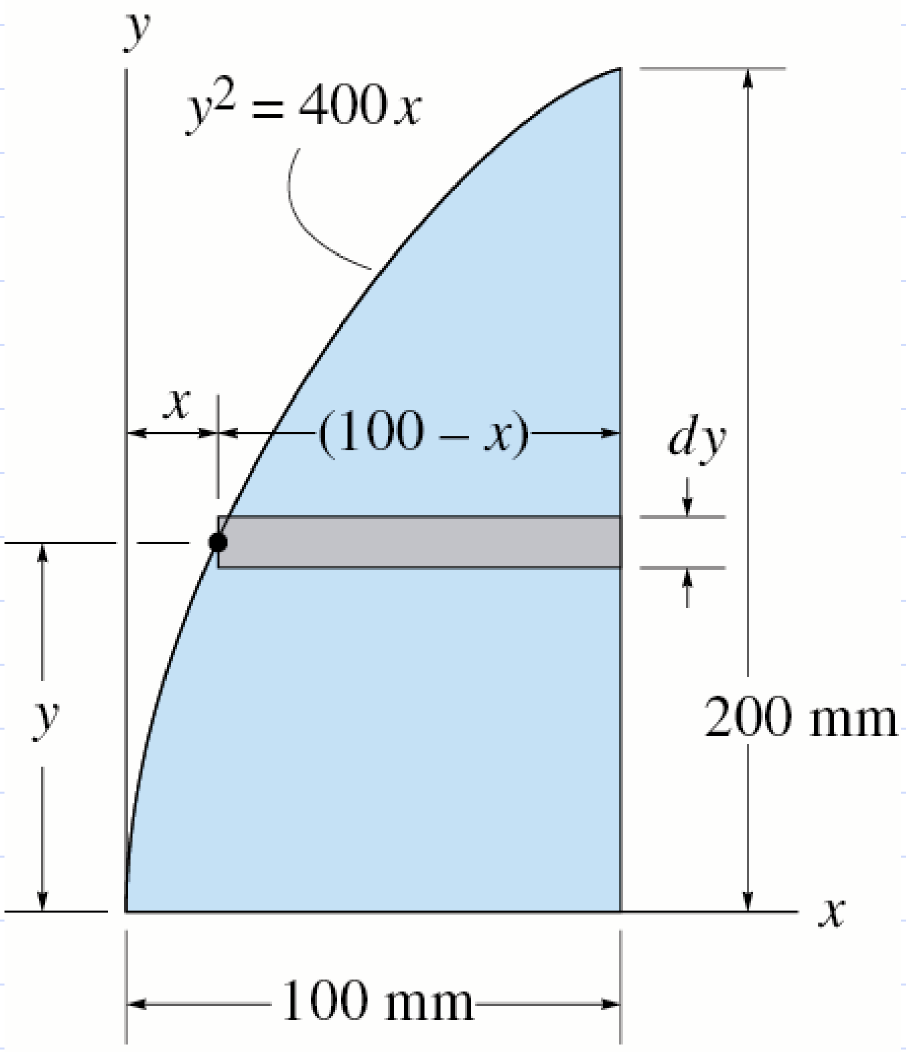


Figure 10.06(a)

$$I_x = \int_A y^2 dA = \int_A y^2 \left(100 - x\right) dy$$

$$I_x = \int_0^{200} y^2 \left(100 - \frac{y^2}{400}\right) dy$$

$$I_x = 100 \int_0^{200} y^2 dy - \frac{1}{400} \int_0^{200} y^4 dx$$

$$I_x = 107 \times 10^6 \text{ mm}^4$$

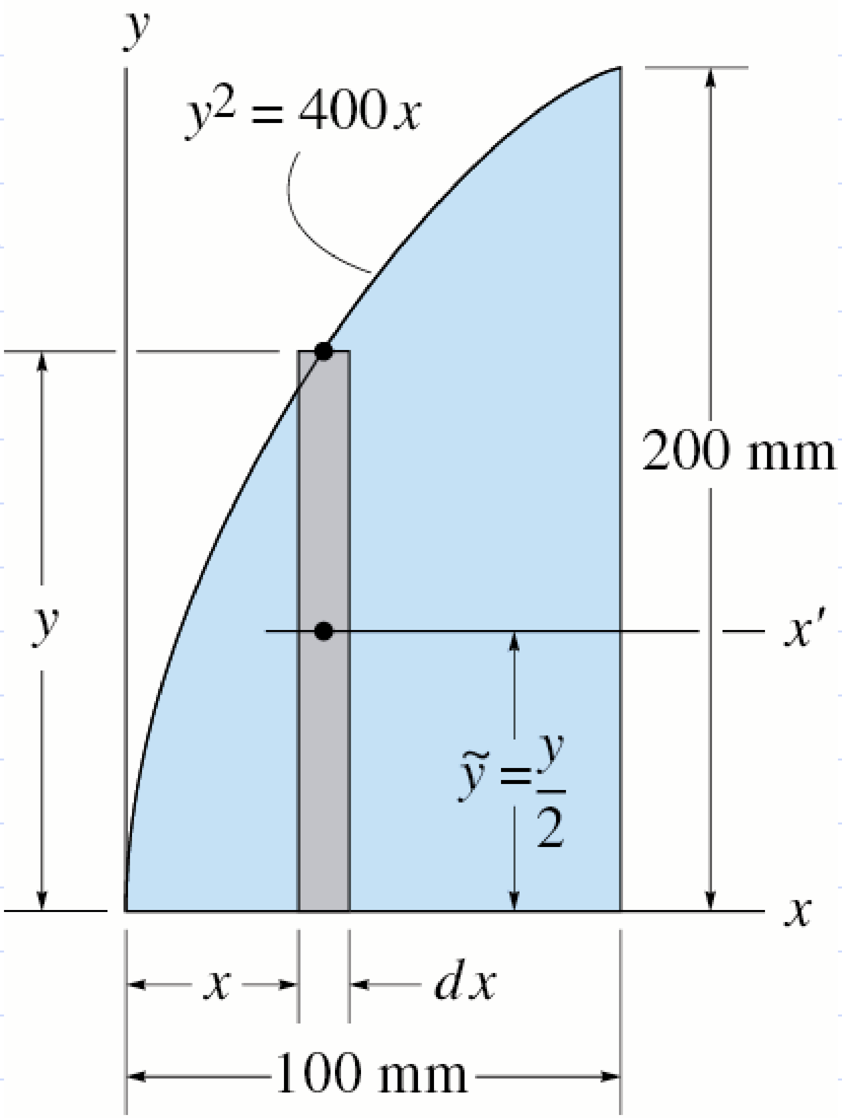


Figure 10.06(b)

$$\bar{I}_{x'} = \frac{1}{12} bh^3$$

$$d\bar{I}_{x'} = \frac{1}{12} dx y^3$$

$$dI_x = d\bar{I}_{x'} + dA\tilde{y}^2$$

$$dI_x = \frac{1}{12} dx y^3 + y dx \tilde{y}^2$$

$$dI_x = \frac{1}{12} dx y^3 + y dx \left(\frac{y}{2}\right)^2$$

$$dI_x = \frac{1}{3} y^3 dx$$

$$dI_x = \frac{1}{3}y^3dx$$

$$I_x = \int_A dI_x = \int_A \frac{1}{3}y^3dx$$

$$I_x = \int_0^{100} \frac{1}{3}(400x)^{3/2}\,dx$$

$$I_x = 107 \times 10^6 \text{ mm}^4$$

PROBLEM

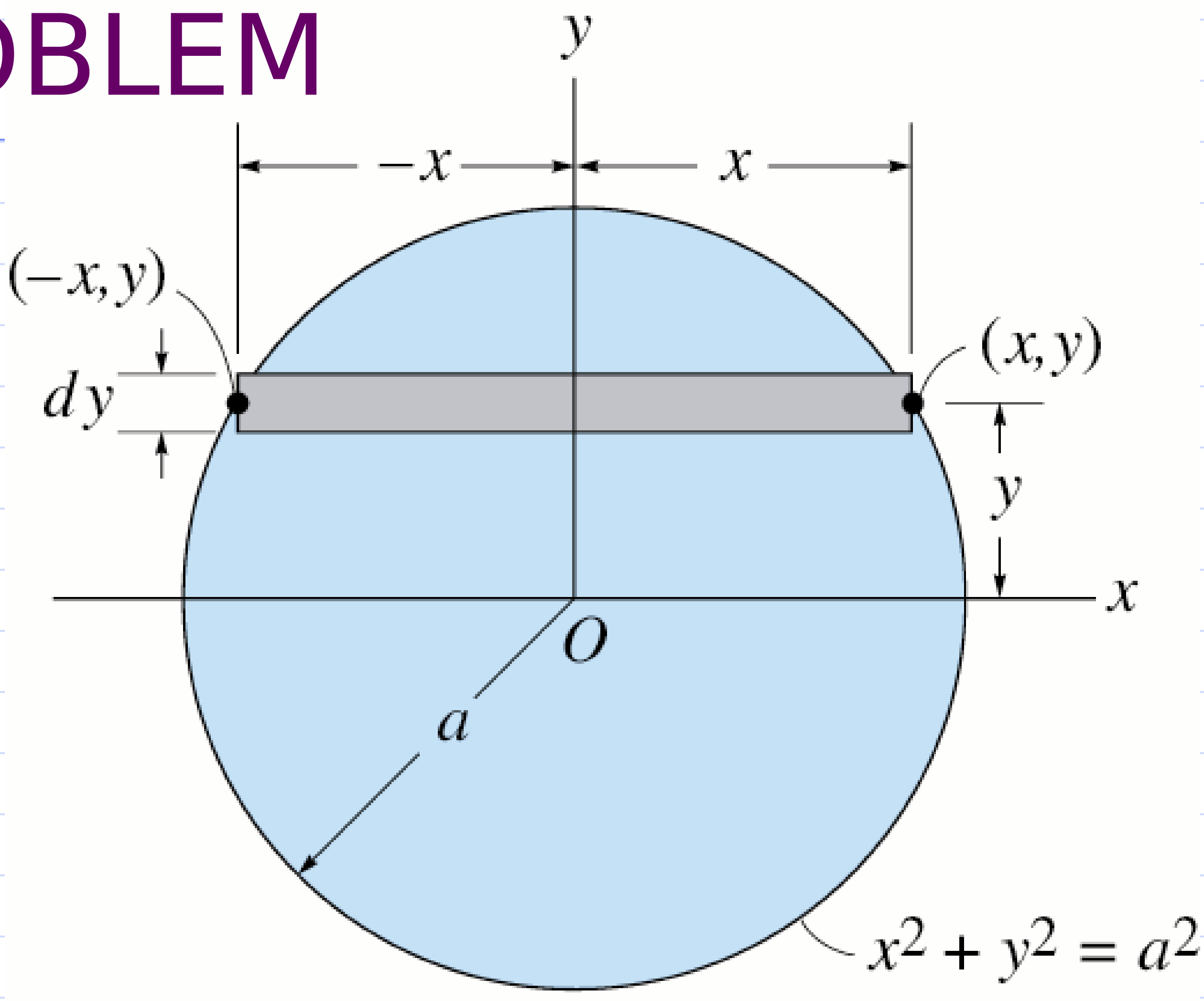


Figure 10.07(a)

$$I_x = \int_A y^2 dA = \int_A y^2 (2x)\, dy$$

$$= \int_{-a}^{a} y^2 \left(2\left(\sqrt{a^2 - x^2}\right)\right) dy = \frac{\pi a^4}{4}$$

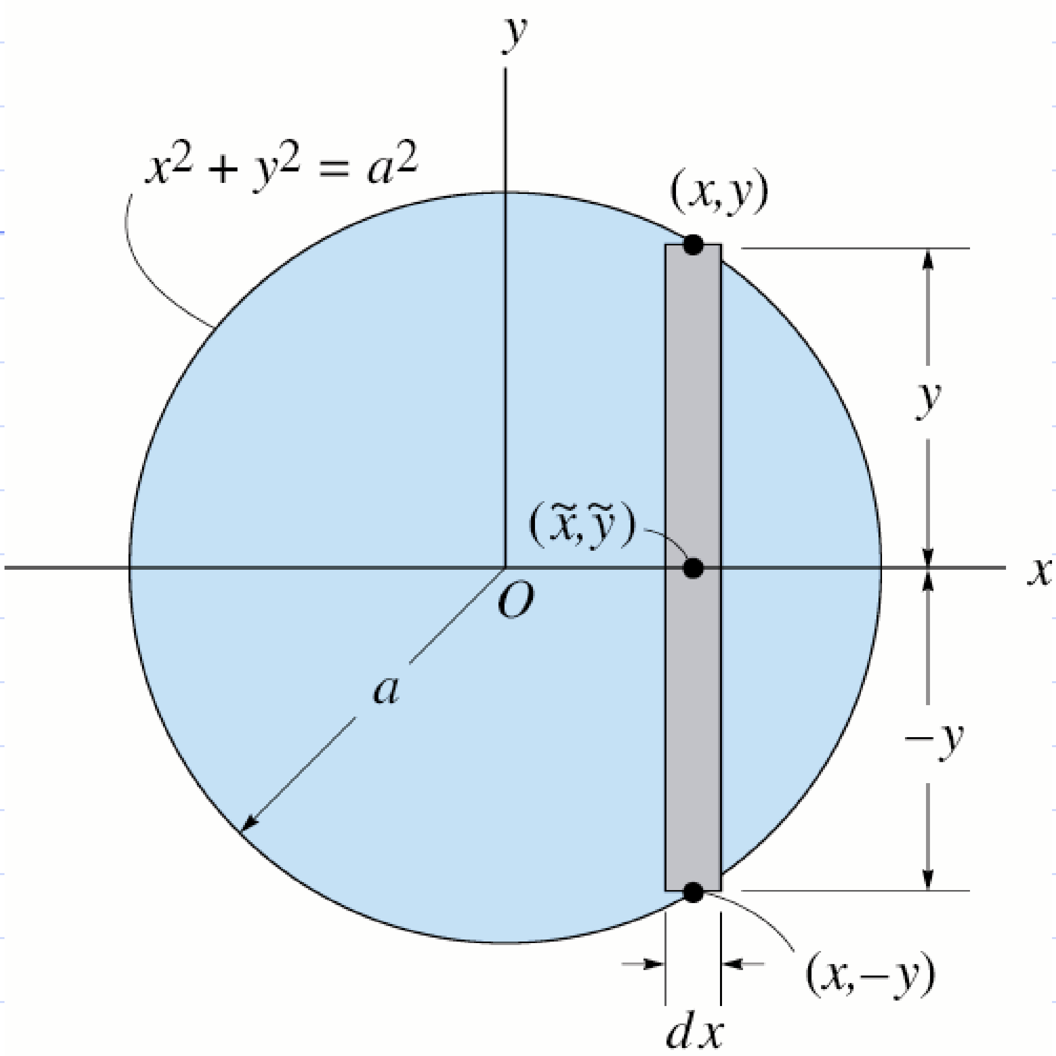


Figure 10.07(b)

$$dI_x = \frac{1}{12} dx \left(2y\right)^3 = y^3 dx$$

$$I_x = \int_{-a}^{a} \frac{2}{3}\left(a^2 - x^2\right)^{\frac{2}{3}} dx = \frac{\pi a^4}{4}$$

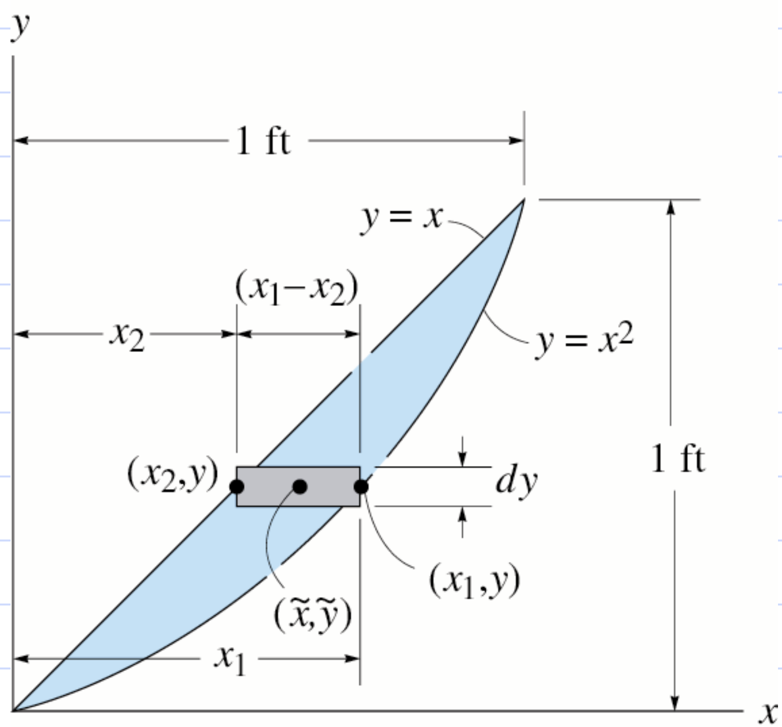


Figure 10.08(a)

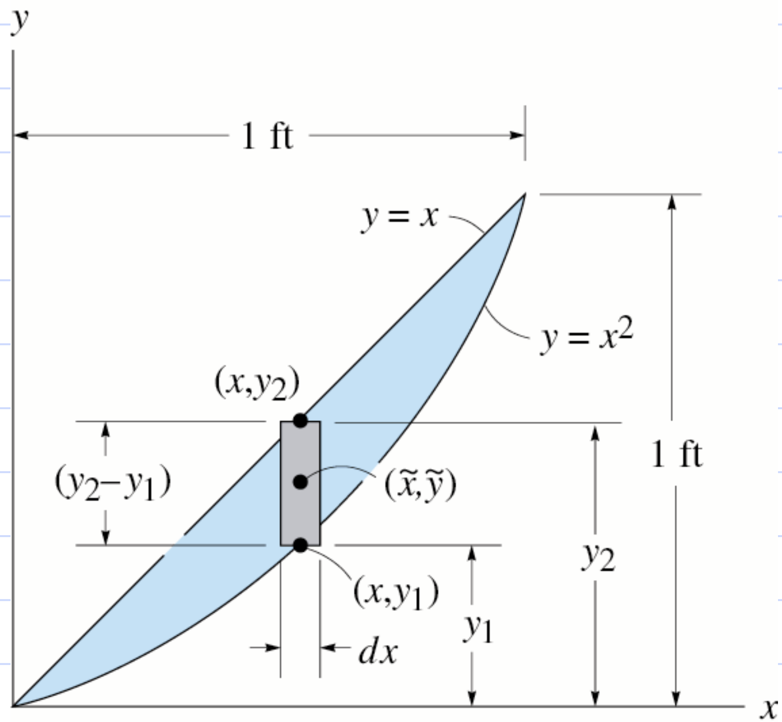


Figure 10.08(b)

Composite Areas

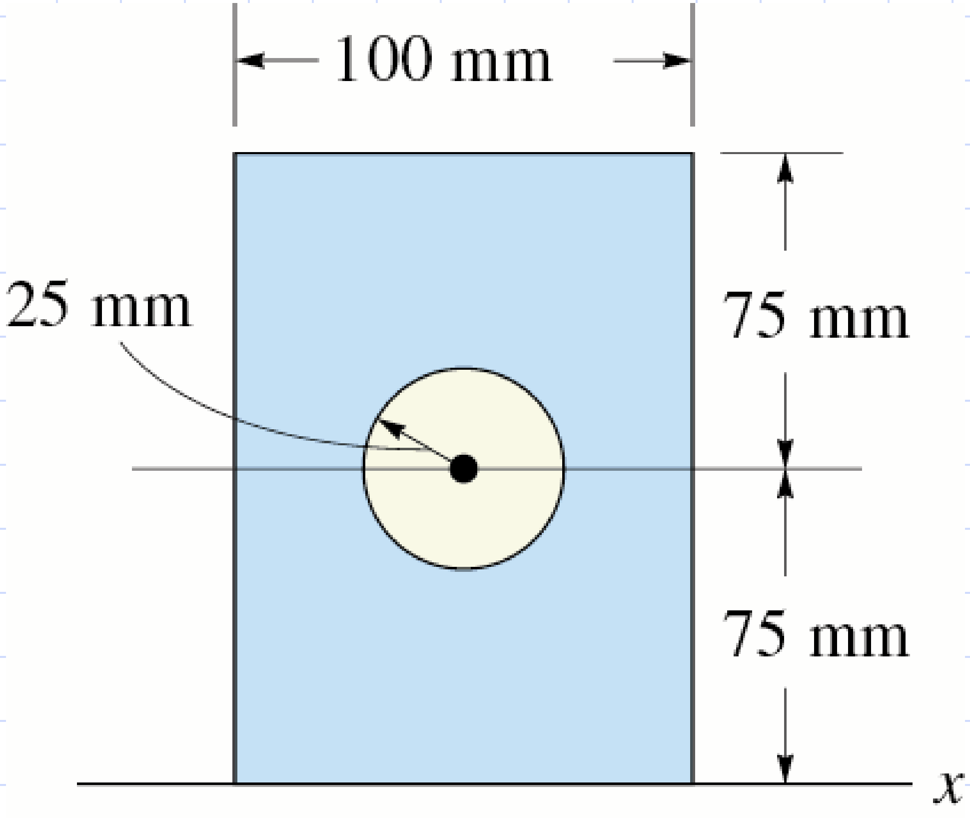


Figure 10.09(a)

Composite Areas

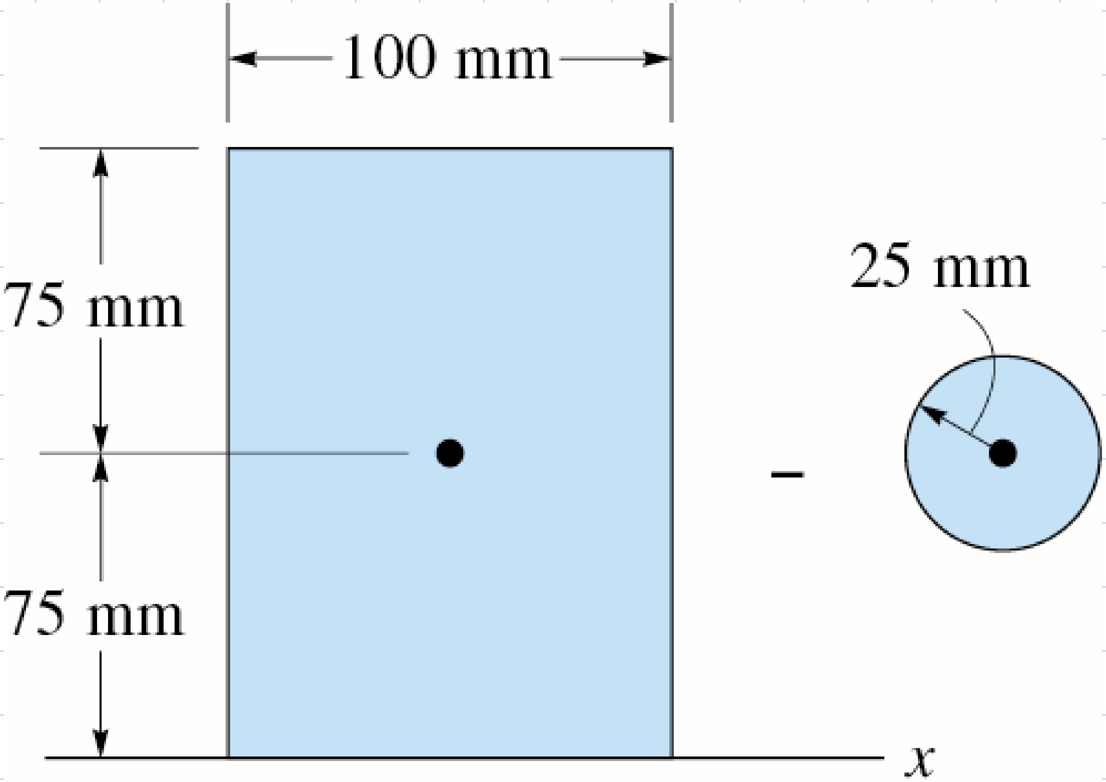
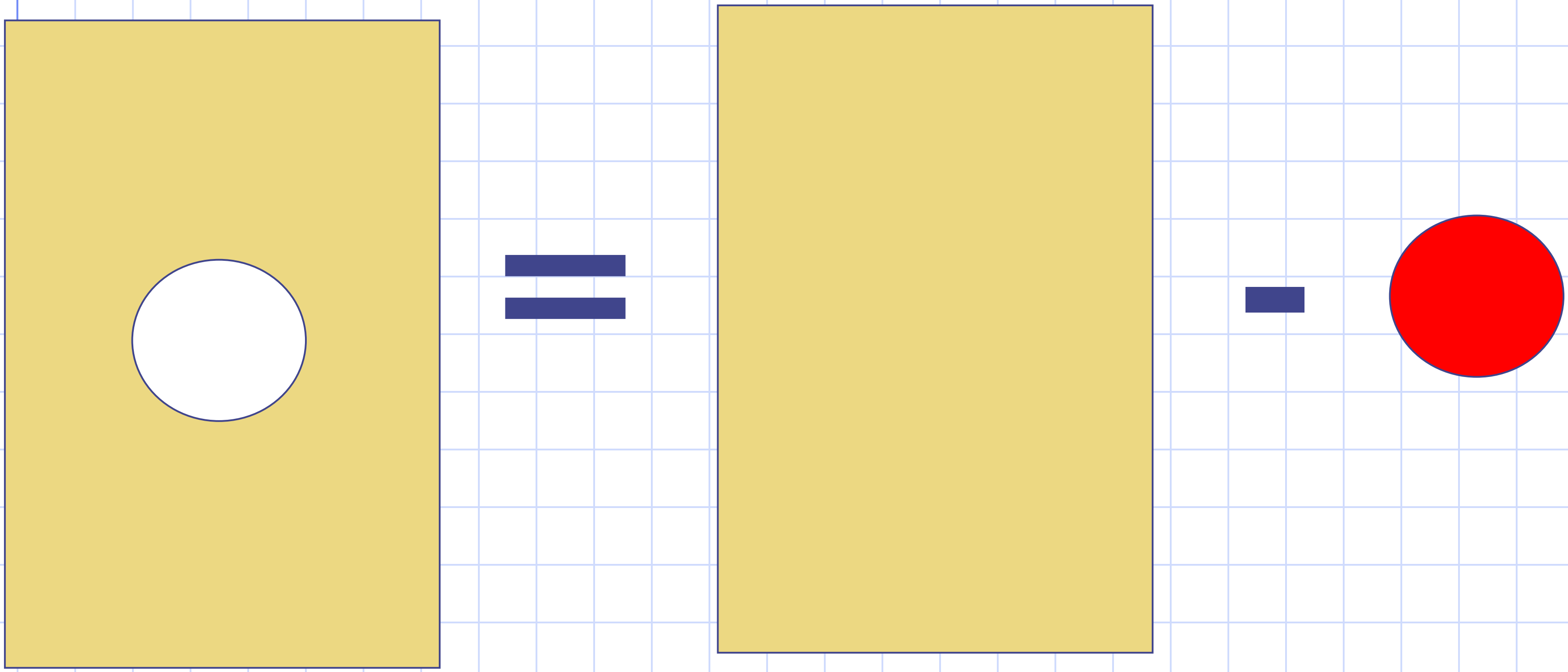
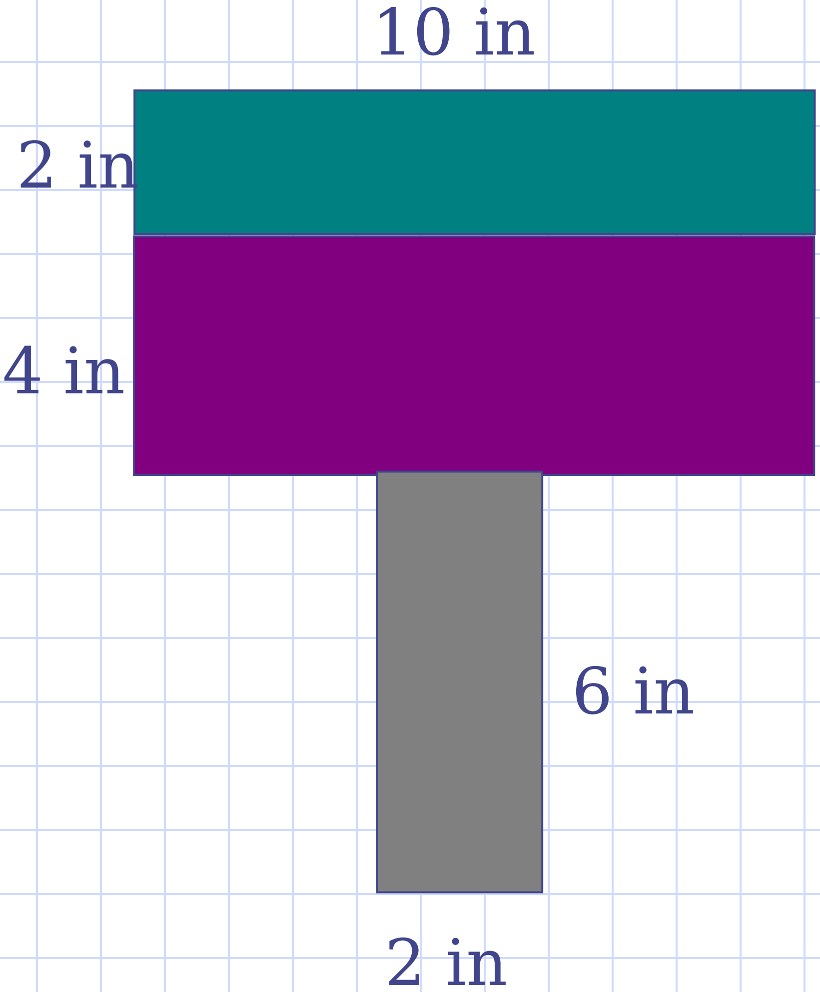


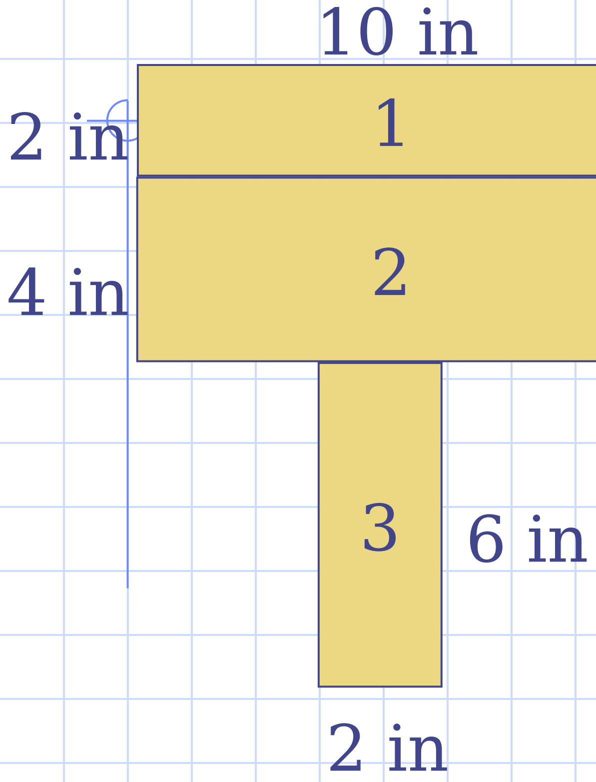
Figure 10.09(b)

Composite Areas





Locate centroid and moments of inertia about centroidal axes



1	20	5	11	100	220
2	40	5	8	200	320
3	12	5	3	60	36
	72			360	576

$$\bar{x} = \frac{360}{72} = 5$$

$$\bar{y} = \frac{576}{72} = 8$$

$$I_x = I_{x1} + I_{x2} + I_{x3}$$

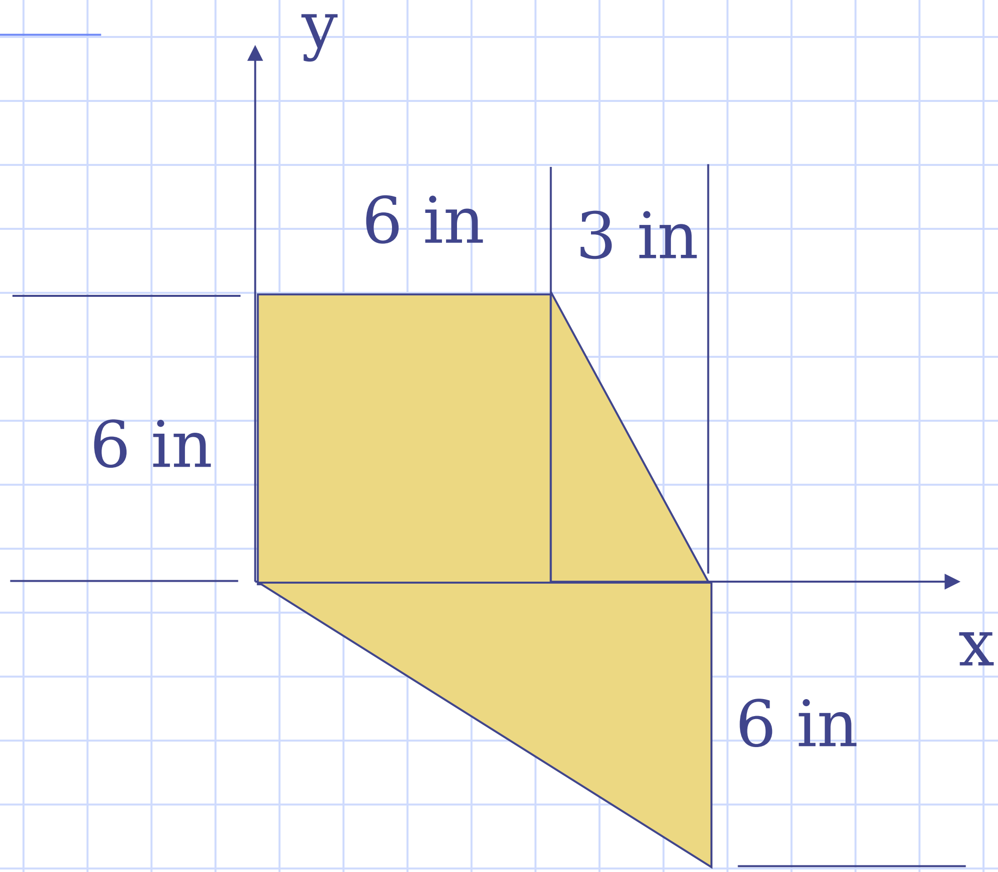
$$I_{x1} = \frac{1}{12}10(2)^3 + 10(2)(11-8)^2 = 186\frac{2}{3}$$

$$I_{x2} = \frac{1}{12}10(4)^3 + 10(4)(8-8)^2 = 53\frac{1}{3}$$

$$I_{x3} = \frac{1}{12}2(6)^3 + 6(2)(8-3)^2 = 336$$

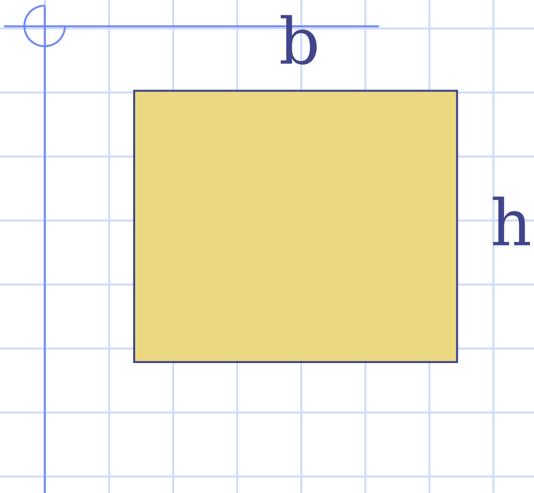
$$I_x = 186\frac{2}{3} + 53\frac{1}{3} + 336 = 576$$

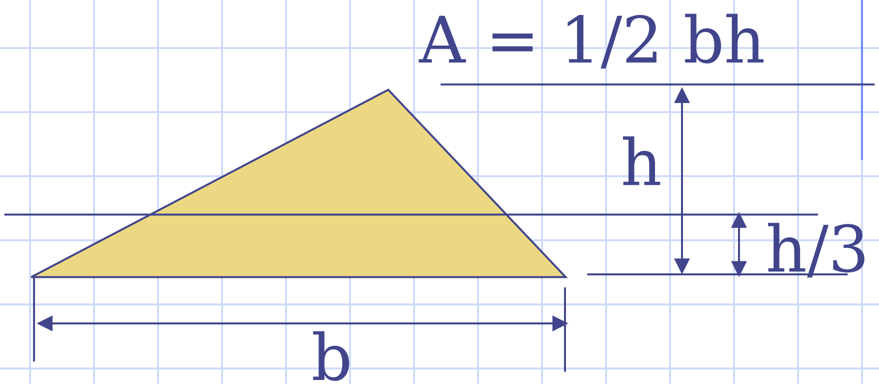
Find I_x and I_y



Region	Area	I_x	I_y	d_x	d_y
1	36	108	108	3	3
2	9	18	45	7	2
3	27	54	121.5	6	2

$$I_y = \left[\frac{1}{12}(b)(h)^3 + (b)(h)\left(\frac{b}{2}\right)^2\right] + \left[\frac{1}{36}(b)(h)^3 + \frac{1}{2}(b)(h)(d)^2\right] + \left[\frac{1}{36}(b)(h)^3 + \frac{1}{2}(b)(h)(d)^2\right]$$





$$I_y = \left[\frac{1}{12}(6)(6)^3 + (6)(6)(3)^2\right]$$

$$+ \left[\frac{1}{36}(6)(3)^3 + \frac{1}{2}(6)(3)(7)^2\right] + \left[\frac{1}{36}(6)(9)^3 + \frac{1}{2}(6)(9)(9)^2\right] =$$

$1971\ in^4$

$$I_y = \left[\frac{1}{12}(6)(6)^3 + (6)(6)(3)^2\right]$$

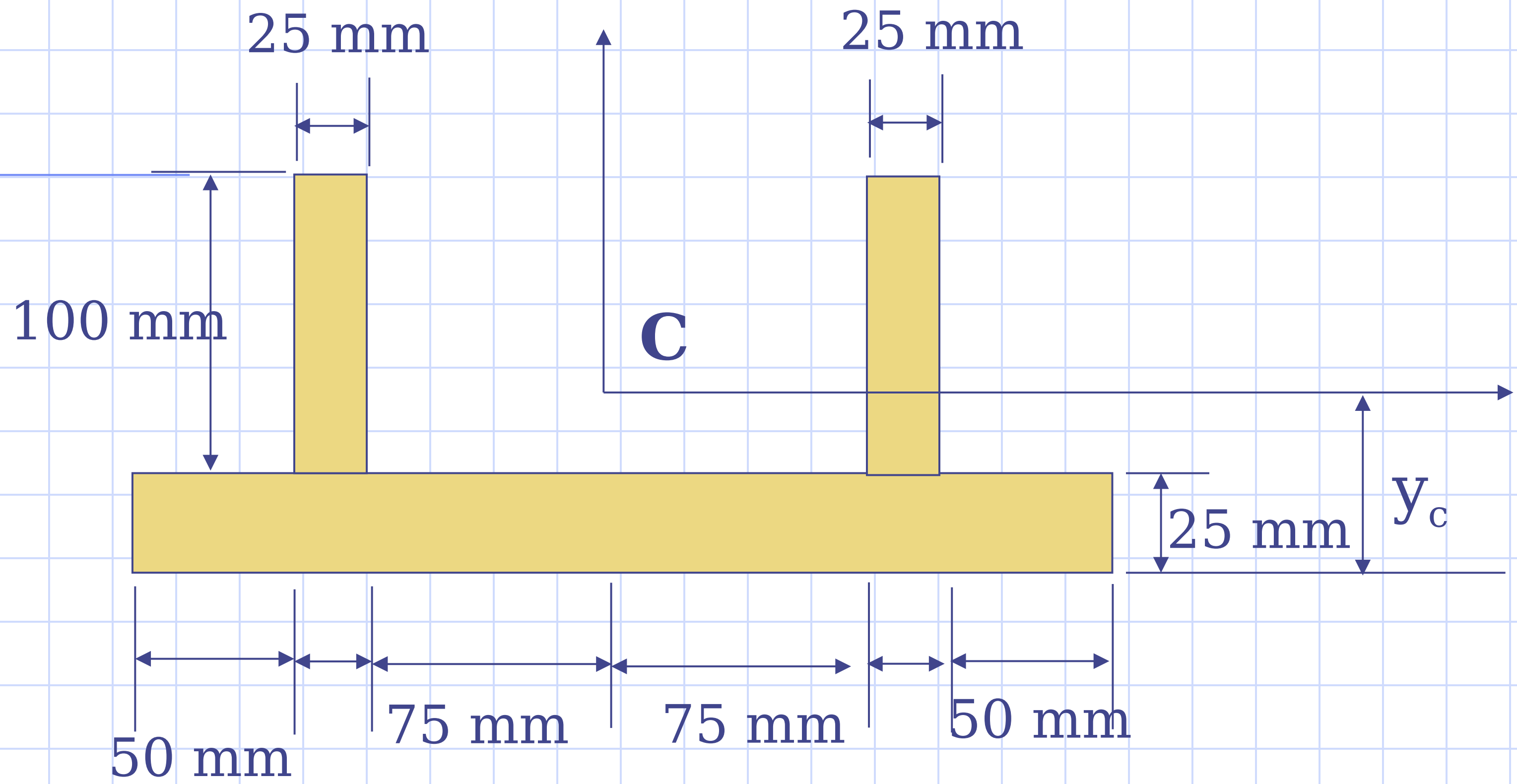
$$+\left[\frac{1}{36}(6)(3)^3 + \frac{1}{2}(6)(3)(7)^2\right] + \left[\frac{1}{36}(6)(9)^3 + \frac{1}{2}(6)(9)(9)^2\right] =$$

$$1971 \text{ in}^4$$

$$I_x = \left[\frac{1}{12}(6)(6)^3 + (6)(6)(3)^2\right]$$

$$+\left[\frac{1}{36}(3)(6)^3 + \frac{1}{2}(6)(3)(2)^2\right] + \left[\frac{1}{36}(9)(6)^3 + \frac{1}{2}(6)(9)(2)^2\right] =$$

$$630 \text{ in}^4$$



Determine I_x and I_y about the centroidal axes.

Region	Area	x	y	xA	yA
1	7500	0	12.5	0	93750
2	2500	-87.5	75	-218750	187500
3	2500	87.5	75	218750	187500
	12500			0	468750

$$\mathbf{y_c = \frac{468750}{12500} = 37.5}$$

$$I_x = \left[\frac{1}{12}(300)(25)^3 + (300)(25)(37.5 - 12.5)^2\right]$$

$$+ \left[\frac{1}{12}(25)(100)^3 + (100)(25)(37.5 - 75)^2\right]$$

$$+ \left[\frac{1}{12}(25)(100)^3 + (100)(25)(37.5 - 75)^2\right]$$

$$I_{\bar{x}} = 16.3 \times 10^6 \ \text{mm}^4$$

$$I_x = \left[\frac{1}{12}(25)(300)^3 + (300)(25)(0)^2\right]$$

$$+ \left[\frac{1}{12}(100)(25)^3 + (100)(25)(0 - 87.5)^2\right]$$

$$+ \left[\frac{1}{12}(100)(25)^3 + (100)(25)(87.5 - 0)^2\right]$$

$$I_{\bar{x}} = 94.8 \times 10^6 \ \text{mm}^4$$

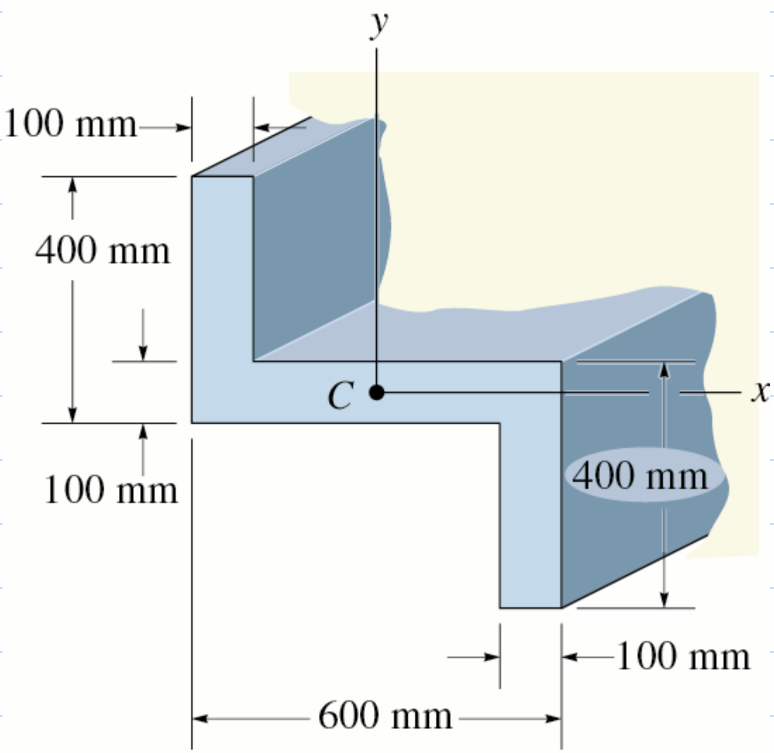


Figure 10.10(a)

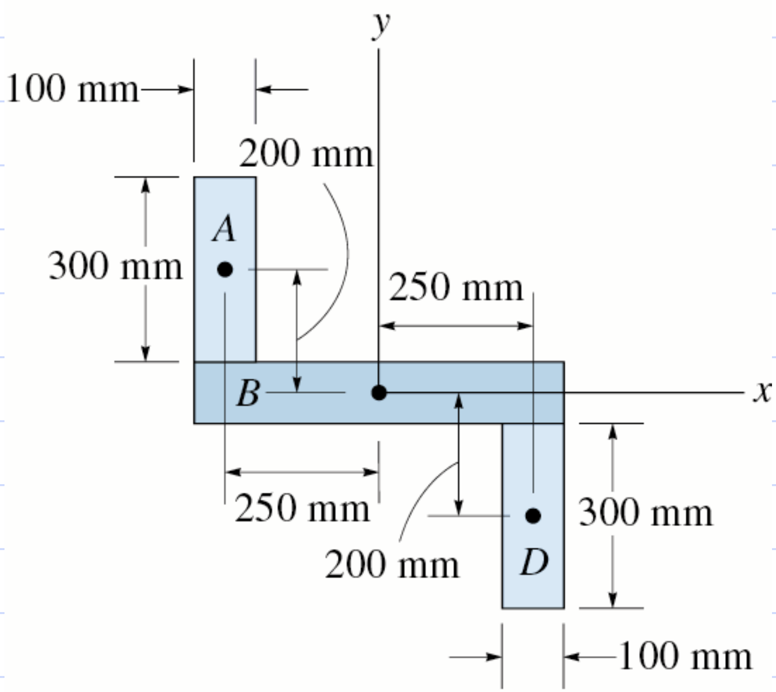


Figure 10.10(b)

Product of Inertia

$$I_{xy} = \int_A xy\,dA$$

Product of Inertia

$$I_{xy} = 0$$

If either x or y is a line of symmetry.

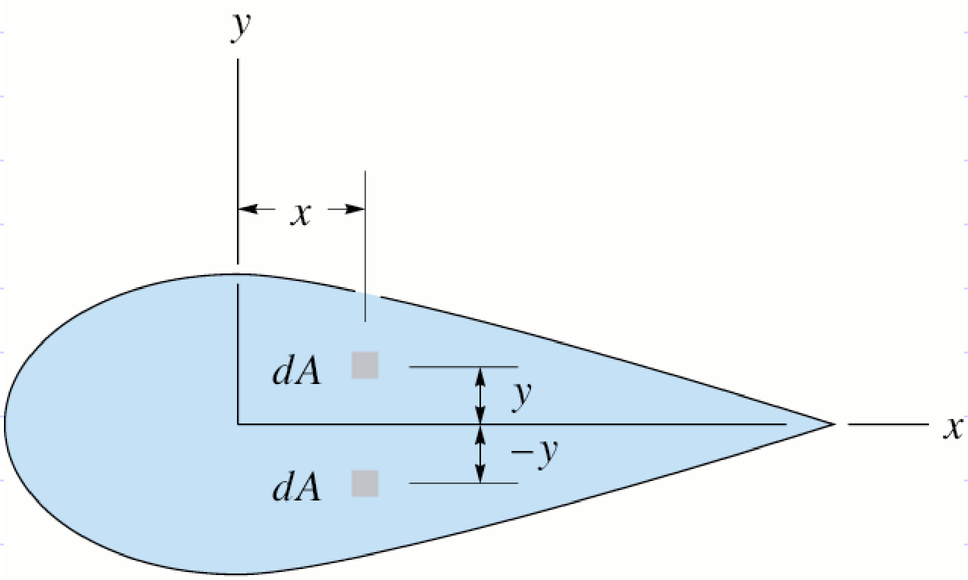
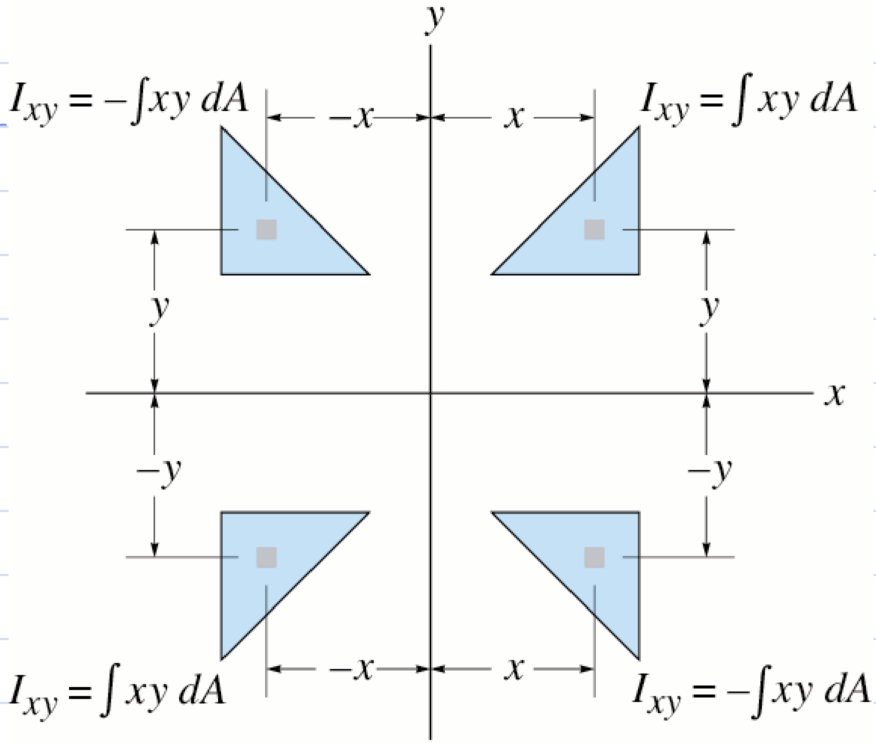


Figure 10.12



y
$I_{xy} = -\int xy\, dA$
$-x$
x
$I_{xy} = \int xy\, dA$
y
y
x
$-y$
$-y$
$I_{xy} = \int xy\, dA$
$-x$
x
$I_{xy} = -\int xy\, dA$

Figure 10.13

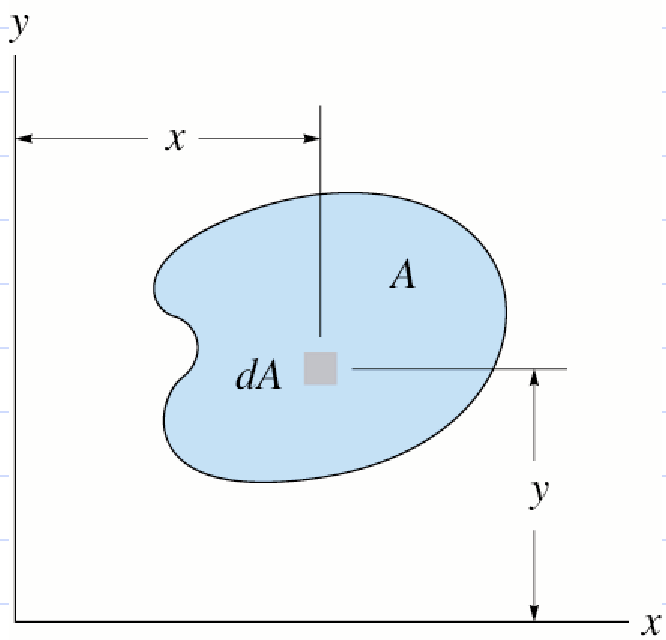


Figure 10.11

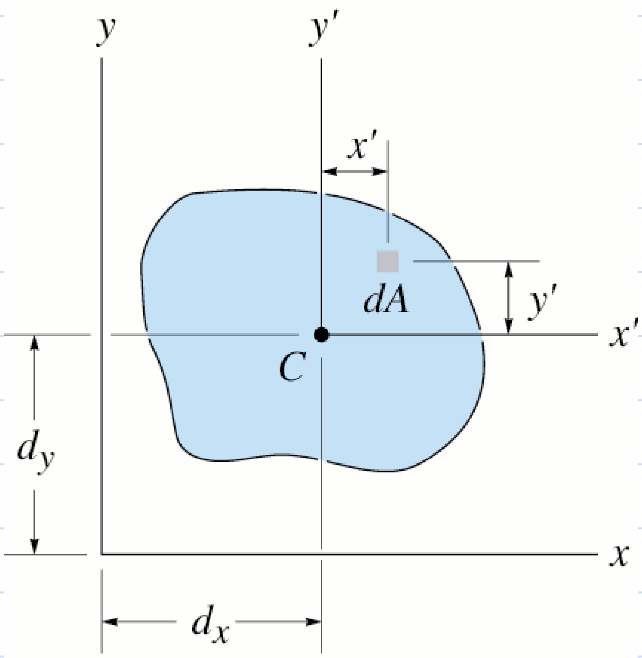


Figure 10.14

Parallel Axis Theorem for Product of Inertia

$$I_{xy} = \bar{I}_{xy} + A d_x d_y$$

Algebraic signs of d_x and d_y are important!

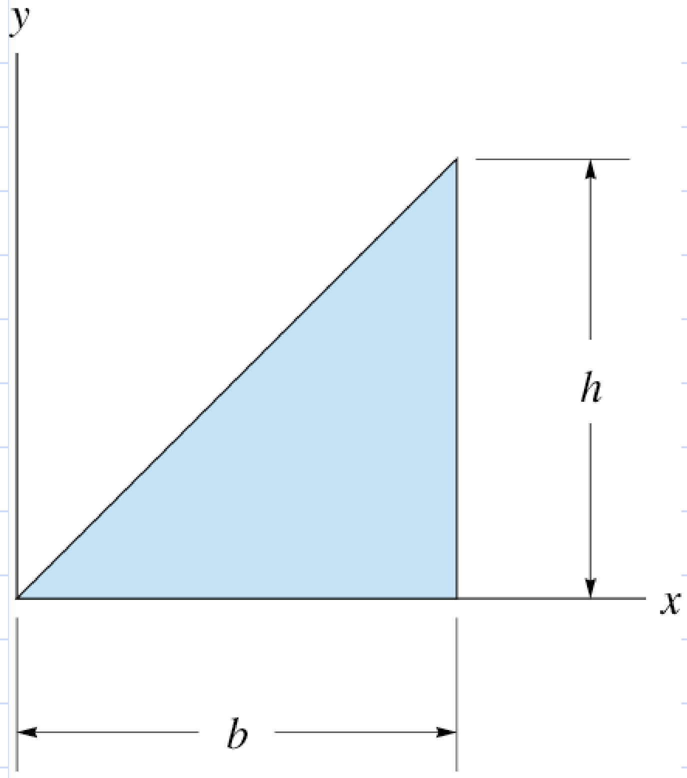


Figure 10.15(a)

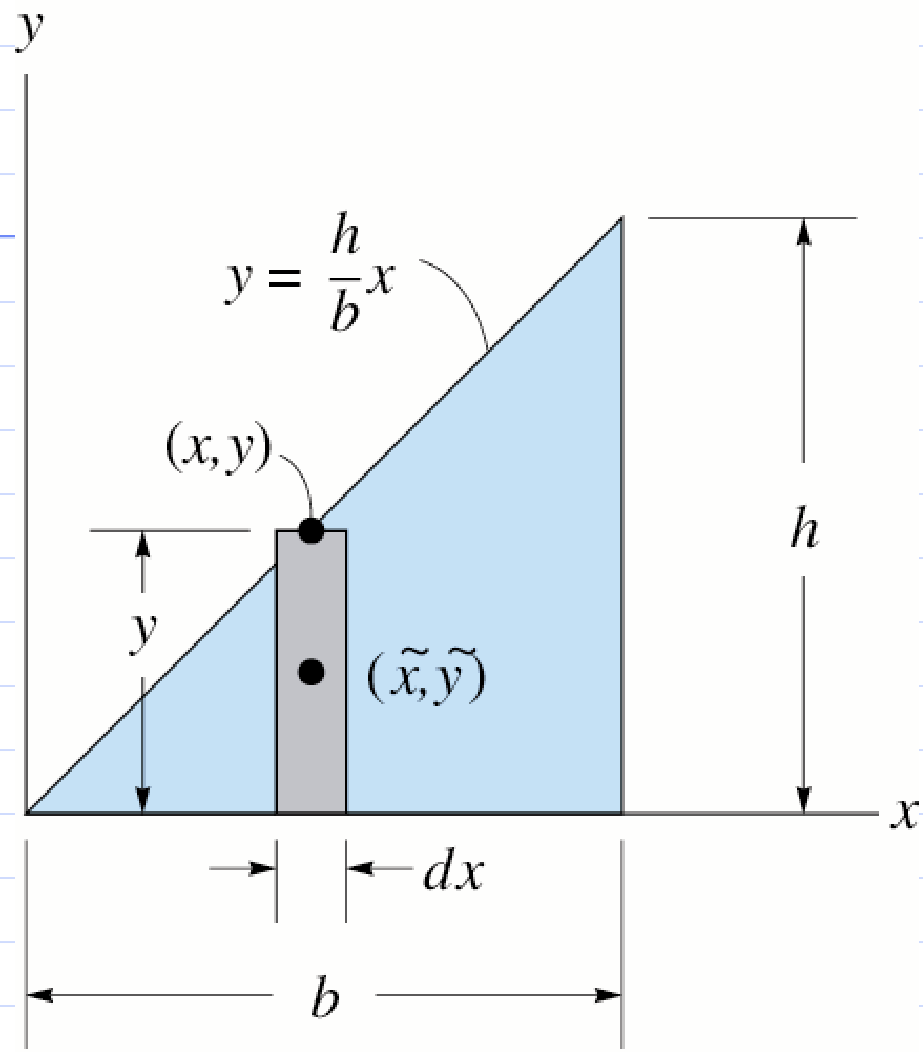


Figure 10.15(b)

$$dI_{xy} = d\bar{I}_{xy} + dA\bar{x}\bar{y}$$

$$dI_{xy} = 0 + (y\,dx)\,x\,\frac{y}{2}$$

$$dI_{xy} = 0 + \left(\frac{h}{b}x\,dx\right)x\,\frac{h}{2b}x$$

$$I_{xy} = \int_0^b \frac{h^2}{2b^2}x^3\,dx$$

$$I_{xy} = \frac{b^2h^2}{8}$$

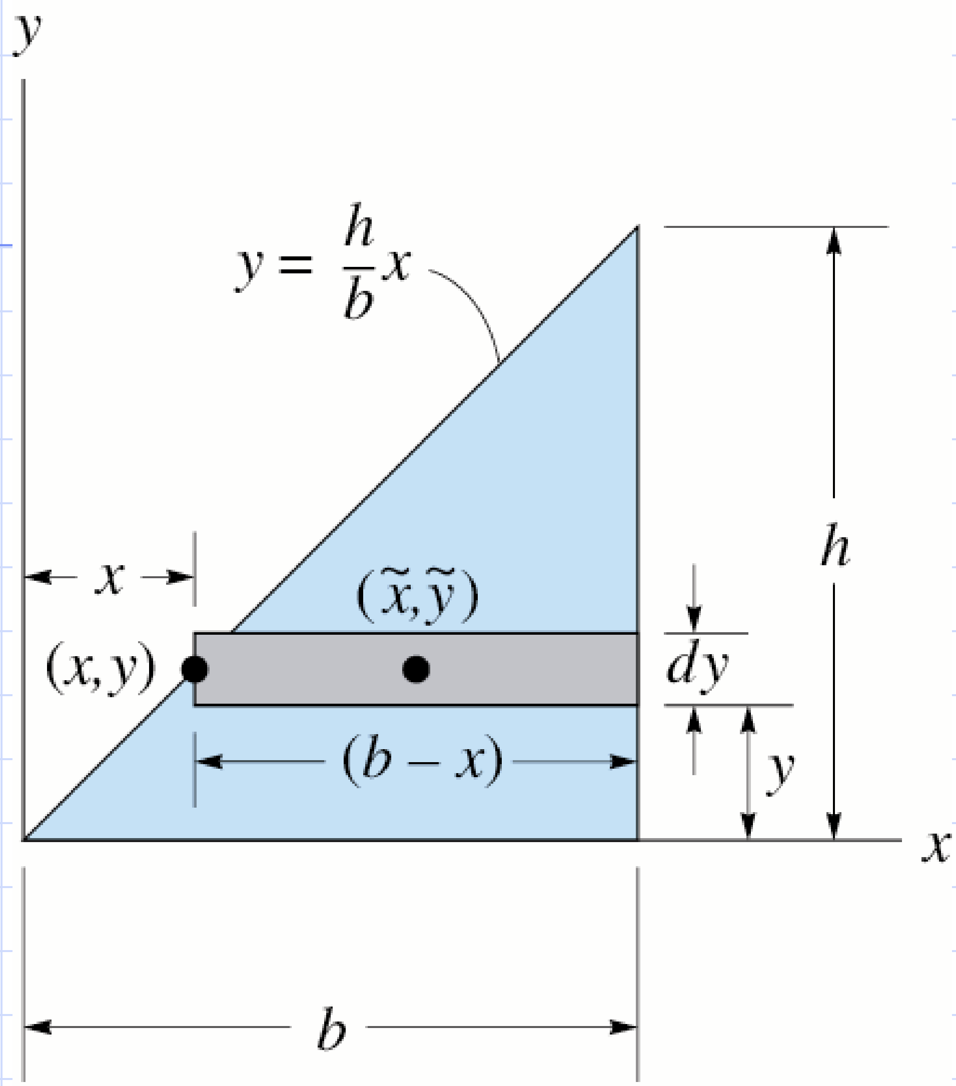


Figure 10.15(c)

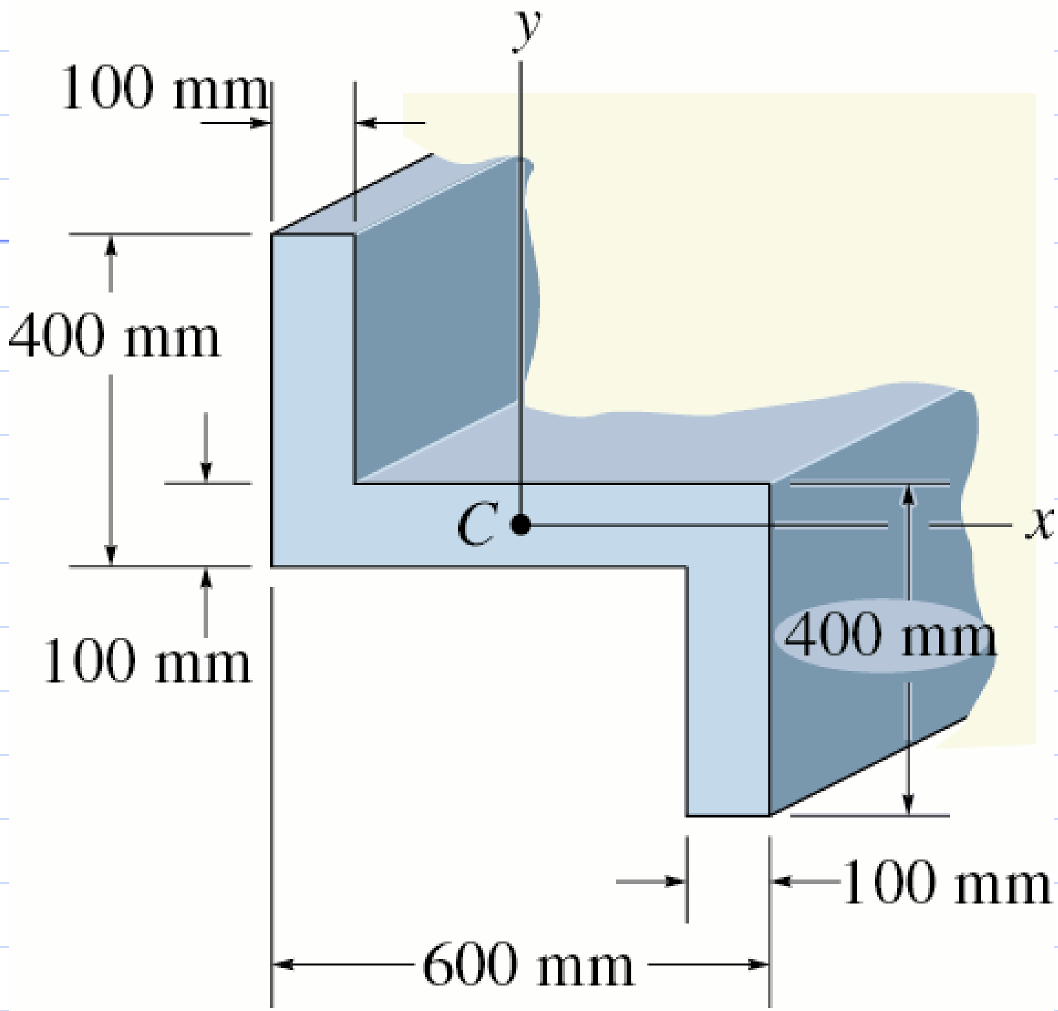
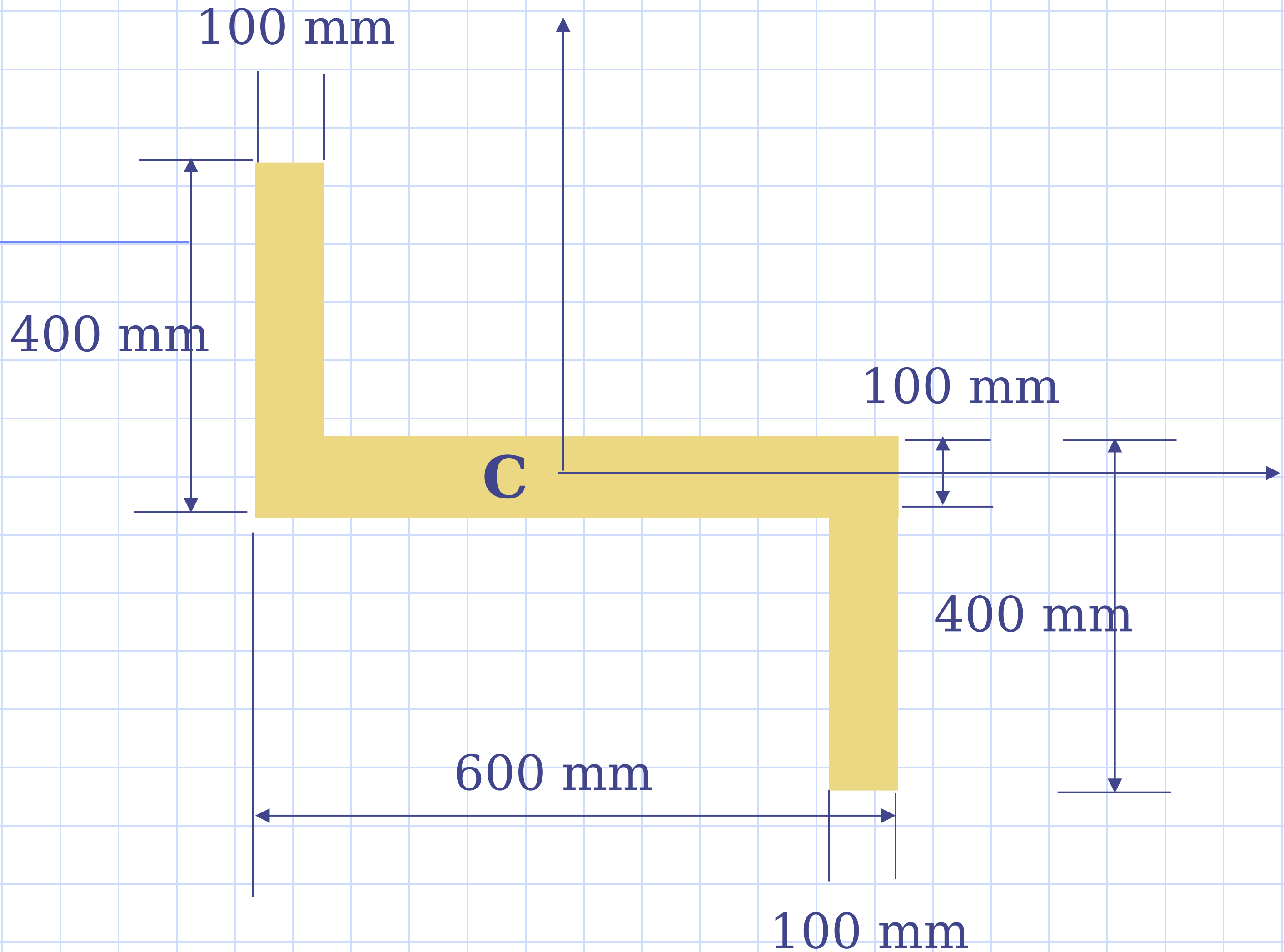
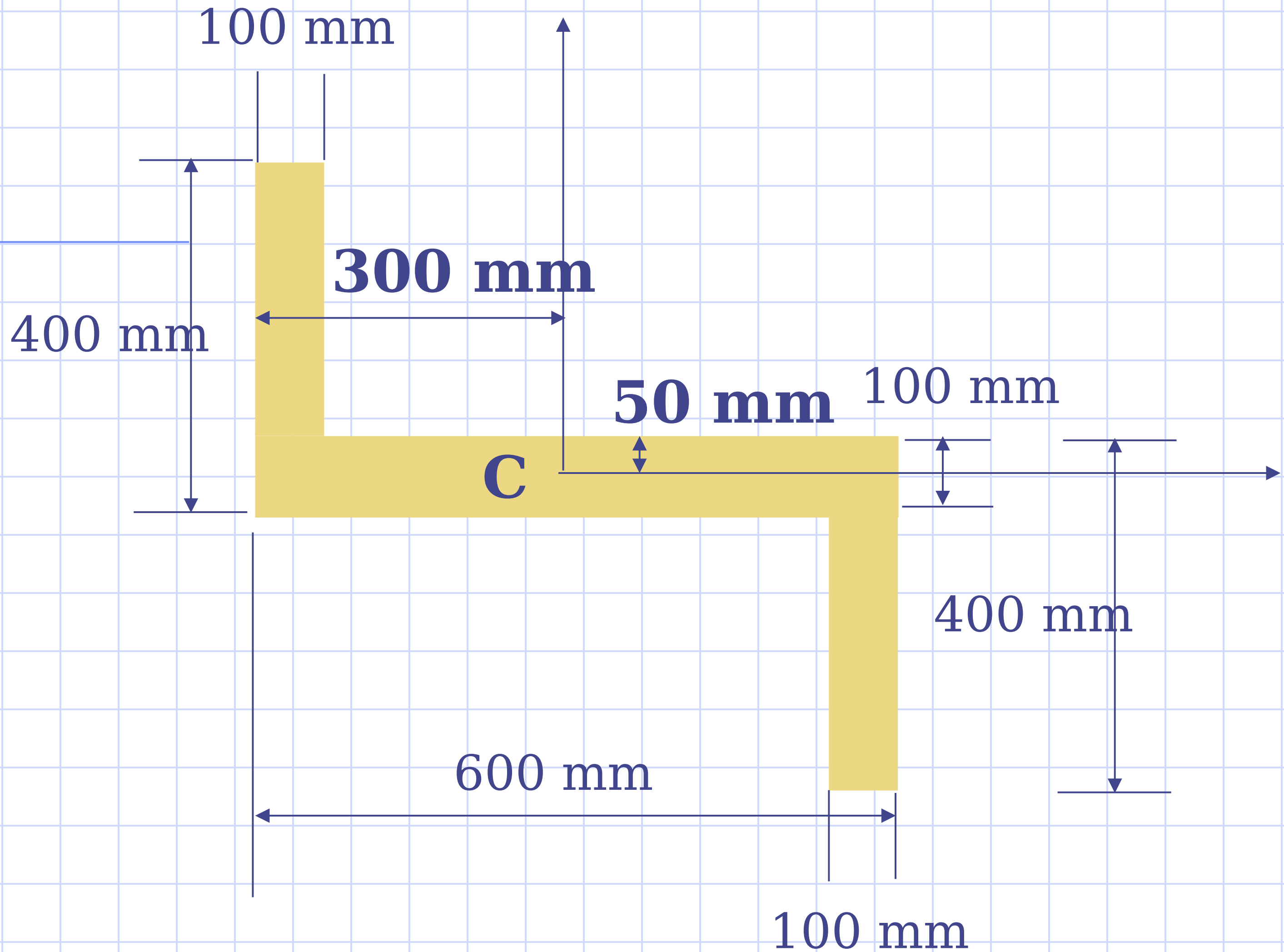


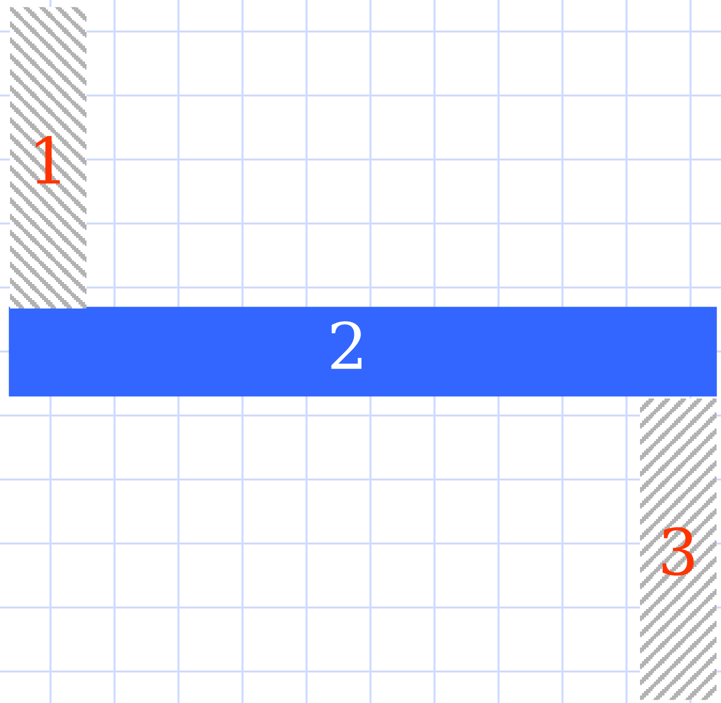
Figure 10.16(a)



Find the product of inertia about the centroidal axes.



Centroid due to symmetry.


1
2
3

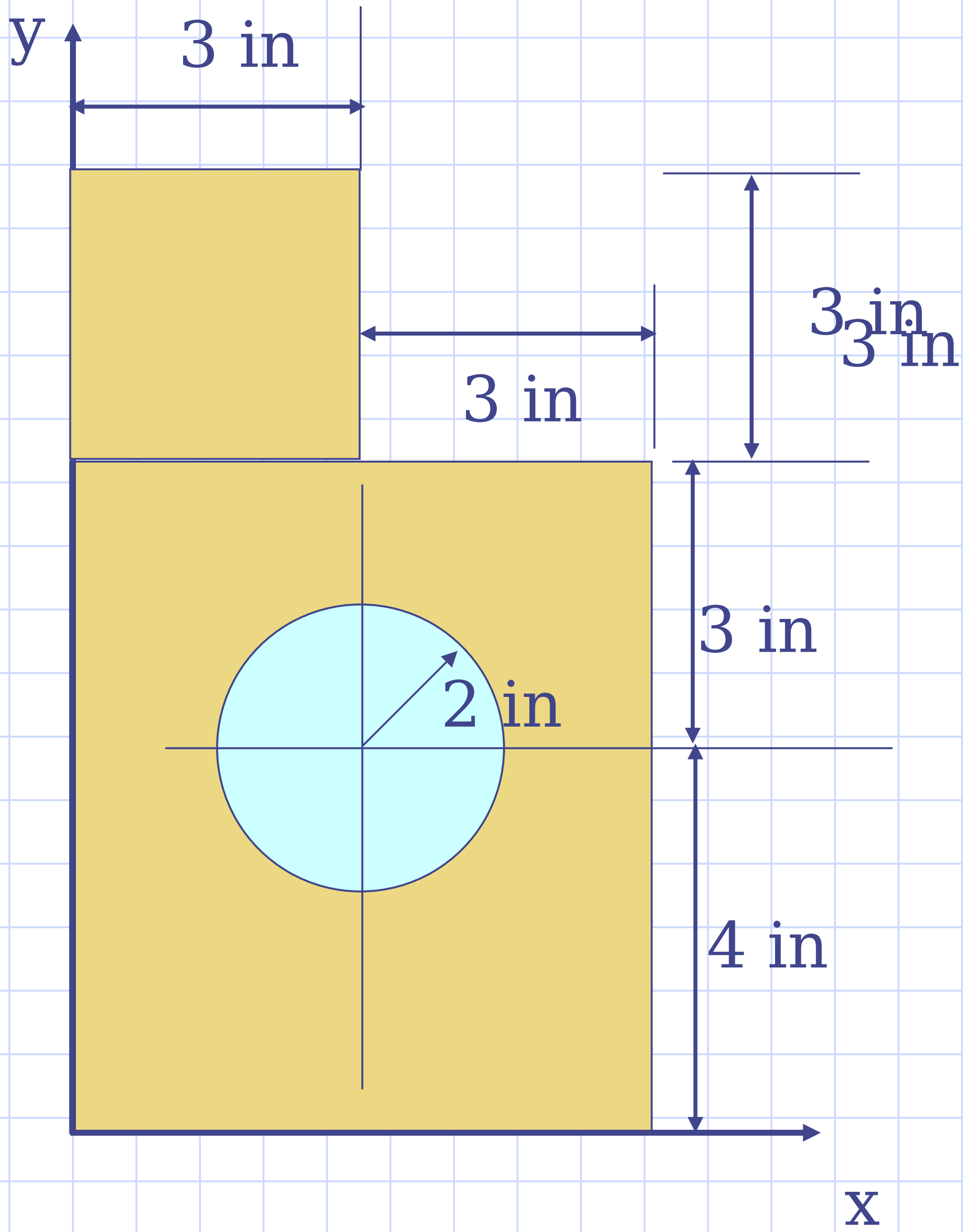
$$I_{xy} = \left[0 + (300)(100)(-250)(200)\right]$$

$$+ \left[0 + (600)(100)(0)(0)\right]$$

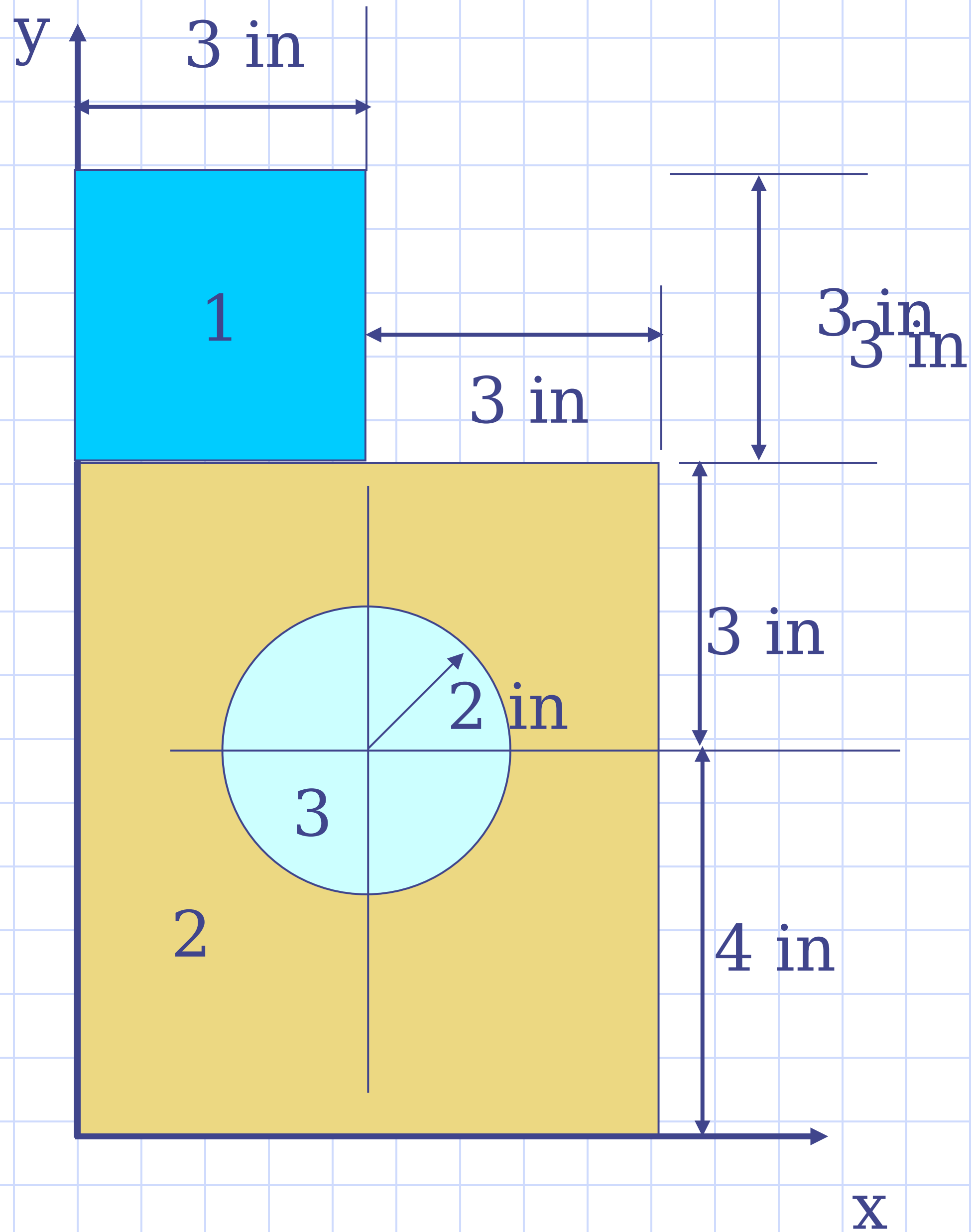
$$+ \left[0 + (300)(100)(250)(-200)\right]$$

$$I_{xy} = -3.00 \times 10^{9}\ \text{mm}^{4}$$

Find I_x, I_y, and I_{xy} about x-y axes and about centroidal axes



Identify sub-areas



Find Centroid Location

Area #	Area	x	y	xA	yA
1	9	1.5	8.5	13.5	76.5
2	42	3	3.5	126	147
3	-12.566	3	4	-37.699	-50.265
	38.434			101.8	173.23
	x=	2.649			
	y=	4.507			

ind Moment of Inertia about x-y Axes

$$I_x = \left(\frac{1}{12} 3(3)^3 + (3)(3)(8.5)^2 \right) + \left(\frac{1}{12} 6(7)^3 + (6)(7)(3.5)^2 \right)$$

$$- \left(\frac{1}{4} \pi (2)^4 + (\pi)(2)^2 (4)^2 \right) = 1129.372$$

$$I_y = \left(\frac{1}{12} 3(3)^3 + (3)(3)(1.5)^2 \right) + \left(\frac{1}{12} 7(6)^3 + (6)(7)(3)^2 \right)$$

$$- \left(\frac{1}{4} \pi (2)^4 + (\pi)(2)^2 (3)^2 \right) = 405.3363$$

ind Moment of Inertia about x-y Axes

$$I_{xy} = \left(0 + (3)(3)(8.5)(1.5)\right)$$

$$+\left(0 + (6)(7)(3.5)(3)\right)$$

$$-\left(0 + (\pi)(2)^2(4)(3)\right)$$

$$= 404.9536$$

nd Moment of Inertia about Centroidal A

$$I_x = \left(\frac{1}{12}3(3)^3 + (3)(3)(8.5 - 4.507)^2\right) + \left(\frac{1}{12}6(7)^3 + (6)(7)(3.5 - 4.507)^2\right)$$

$$- \left(\frac{1}{4}\pi(2)^4 + (\pi)(2)^2(4 - 4.507)^2\right) = 348.54$$

$$I_y = \left(\frac{1}{12}3(3)^3 + (3)(3)(1.5 - 2.649)^2\right) + \left(\frac{1}{12}7(6)^3 + (6)(7)(3 - 2.649)^2\right)$$

$$- \left(\frac{1}{4}\pi(2)^4 + (\pi)(2)^2(3 - 2.649)^2\right) = 135.6917$$

ind Moment of Inertia about x-y Axes

$$I_{xy} = (0 + (3)(3)(8.5 - 4.507)(1.5 - 2.649))$$
$$+ (0 + (6)(7)(3.5 - 4.507)(3 - 2.649))$$
$$- \left(0 + (\pi)(2)^2(4 - 4.507)(3 - 2.649)\right) = -49.486$$